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EXCESS CLAIMS AND DATA TRIMMING IN THE CONTEXT OF CREDIBILITY R--ETC(U)
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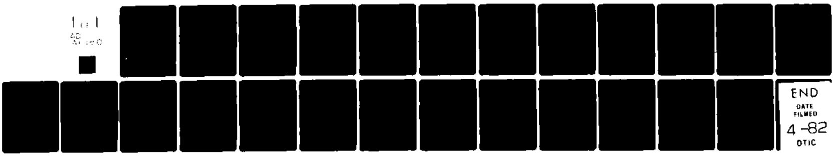

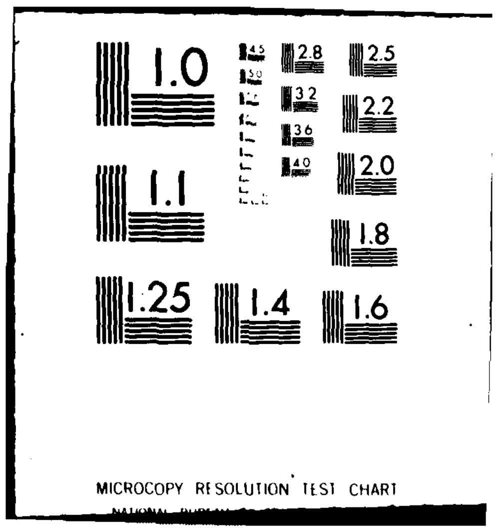
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Excess Claims and Data Trimming in the Context of Credibility Rating Procedures

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Excess Claims and Data Trimming in the Context of Credibility Rating Procedures

by Hans Bühlmann, Alois Gisler, William S. Jewell*

1. Motivation

In Ratemaking and in Experience Rating one is often confronted with the dilemma of whether or not to fully charge very large claims to the claims load of small risk groups or of individual risks. Practitioners typically use an a posteriori argument in this situation: "If such large claims should be fully charged then the rates obtained would become 'ridiculous', hence it should not be done." The present paper aims at explaining this practical attitude from first principles.

Credibility Theory in its standard form makes the first step in the good direction. It explains to us that all claims should not be fully charged (but only with the constant fraction of the credibility weight). In many applications, however, it is still felt that the fraction of this charge should depend on the size of a claim. This leads very naturally to the idea of combining credibility procedures and data trimming.

Of course, such an idea needs to be tested. The first argument in favour of it was given by Gisler [1] who showed that in many cases the mean quadratic loss of the credibility estimator is substantially reduced if one introduces trimming of claims data. This paper goes even further. It formalizes the standard way of thinking about large claims and then shows that "optimal forecasting" of rates (using Bayes estimation techniques) and forecasting by "credibility techniques combined with data trimming" lead to almost identical results.

* The authors are greatly indebted to R. Schnieper who did all the numerical work on the ETH computer.

2. The Basic Model

Throughout the paper we work with the most simple model in the credibility context

$\underline{X} = (X_1, X_2, \ldots, X_n)$ is the random vector representing the experience of a given risk in the years $1, 2, \ldots, n$

- The quality of the risk is characterized by an unknown parameter value θ, which we consider as a realisation of a random variable Θ with distribution function $U(\theta)$
- Given the parameter value θ, $\{X_1, X_2, \ldots, X_n\}$ are i.i.d. with density function $f_\theta(x)$ [mean $\mu(\theta)$, variance $\sigma^2(\theta)$]

To these standard assumption in credibility theory we add now some more structure regarding the distribution of the size of a claim. The main idea is introduced by the assumption that the claims sizes are drawn from two different urns (distributions). Mostly, i.e. with probability $1-\pi$, we observe an ordinary claim with density $p_o(x/_\theta)$ [mean $\mu_o(\theta)$, variance $\sigma_o^2(\theta)$] and occasionally, i.e. with probability π, we observe an excess claim (catastrophic claim) with density $p_e(x/_\theta)$ [mean $\mu_e(\theta)$, variance $\sigma_e^2(\theta)$] .

	$p_o(x/_\theta)$ ordinary claim amounts	$p_e(x/_\theta)$ excess claim amounts
occurrence	$1-\pi$	π

We have assumed that the mixing probabilities are independent of θ and from now on we shall also suppose that the density of the excess claims is independent of the risk parameter, hence formalizing the idea that large catastrophic claims have no bearing on the quality of the risk.

In mathematical shorthand all the considerations just made regarding additional structure are summed up by stating that the density

$f_\theta(x)$ has the following form

1) $f_\theta(x) = (1-\pi)p_o(x_{/\theta}) + \pi p_e(x)$

3. The Basic Problem

As always in the credibility context our aim is to estimate

$\mu(\theta)$	based on the observations of	$\underline{X} = (X_1, X_2, \ldots, X_n)$
pure premium for the given risk		experience of the given risk in the years $1,2,\ldots,n$

One knows that the best estimator for this problem is

$$P[\underline{X}] = E[\mu(\theta)_{/\underline{X}}]$$

Using the special structure of formula 1) we obtain

2) $$P[\underline{X}] = \pi\mu_e + (1-\pi)\underbrace{E[\mu_o(\theta)_{/\underline{X}}]}_{g(\underline{X})}$$

If we use standard credibility techniques we estimate by

3) $$f[\underline{X}] = a + b\sum_{i=1}^{n} X_i \quad \text{with optimal choice of } a,b$$

And if in addition we introduce trimming of the data we estimate by

4) $$f[\underline{X}] = a + b\sum_{i=1}^{n}(X_i \wedge M) \quad \text{with optimal choice of } a,b,M$$

Using 4) we are committing the following error against optimal estimation

$$6)\quad \inf_{a,b,M} E\left\{P[\underline{X}] - f[\underline{X}]\right\}^2 = \inf_{a,b,M} E\left\{\pi\mu_e + (1-\pi)g(\underline{X}) - a - b\sum_{i=1}^{n}(X_i \wedge M)\right\}^2$$

$$= (1-\pi)^2 \inf_{a,b,M} E\left\{\frac{\pi\mu_e - a}{1-\pi} + g(\underline{X}) - \frac{b}{1-\pi}\sum_{i=1}^{n}(X_i \wedge M)\right\}^2$$

$$= (1-\pi)^2 \inf_{a',b',M} E\left\{g(\underline{X}) - a' - b'\sum_{i=1}^{n}(X_i \wedge M)\right\}^2$$

The following two problems are therefore equivalent

A) Estimate $P[\underline{X}]$ (total premium) by $a + b\sum_{i=1}^{n}(X_i \wedge M)$ with optimal a,b,M

B) Estimate $g(\underline{X})$ (ordinary premium) by $a' + b'\sum_{i=1}^{n}(X_i \wedge M)$ with optimal a',b',M

For the optimal choices of the parameters (denoted by ~) we have

$$7)\quad \tilde{a} = (1-\pi)\,\tilde{a}' + \pi\mu_e$$

$$\tilde{b} = (1-\pi)\,\tilde{b}'$$

In the following we want to illustrate that $\tilde{a}' + \tilde{b}'\sum_{i=1}^{n}(X_i \wedge \tilde{M})$ is a good approximation of $g(\underline{X}) = E\,[\mu_o(\theta)/\underline{X}]$ (Problem B) above)

We actually shall compare

$\tilde{a}' + \tilde{b}'\sum(x_i \wedge \tilde{M})$ with $g(\underline{x})$ for any observation $\underline{x}$ of $\underline{X}$

4. The Exact Form of $g(\underline{x})$

Writing out the conditional expectation $E\left[\mu_o(\theta)/\underline{X}=\underline{x}\right]$ we obtain

$$8)\quad g(\underline{x}) = \frac{\int \mu_o(\theta)\left[\prod_{i=1}^{n}\left\{(1-\pi)p_o(x_{i/\theta}) + \pi p_e(x_i)\right\}\right] dU(\theta)}{\int\left[\prod_{i=1}^{n}\left\{(1-\pi)p_o(x_{i/\theta}) + \pi p_e(x_i)\right\}\right] dU(\theta)}$$

Putting $I = \{1,2,\dots n\}$ and $S \subset I$ we rewrite

$$9)\quad \prod_{i=1}^{n}\left\{(1-\pi)p_o(x_{i/\theta}) + \pi p_e(x_i)\right\} = \sum_{S\subset I}(1-\pi)^s\,\pi^{n-s}\prod_{i\in S}p_o(x_{i/\theta})\prod_{i\in\bar{S}}p_e(x_i)$$

where the sum on the right side must be taken over all subsets

$S \subset I$ (including $\emptyset$ and I) with $s = |S|$
and $n = |I|$

We also use the abbreviations

$$p_o(x_S) = \int \prod_{i\in S} p_o(x_{i/\theta})\, dU(\theta)$$

$$p_e(x_S) = \int \prod_{i\in S} p_e(x_i)\, dU(\theta) = \prod_{i\in S} p_e(x_i)$$

$$L\,(x_S) = \left(\frac{\pi}{1-\pi}\right)^{n-s}\frac{p_o(x_S)p_e(x_{\bar{S}})}{p_o(x_I)} \qquad \left[p_o(x_\emptyset) = 1\right]$$

$$E_o\left[\mu_o(\theta)/x_S\right] = \frac{\int \mu_o(\theta)\prod_{i\in S} p_o(x_{i/\theta})\, dU(\theta)}{p_o(x_S)}$$

Then introducing 9) into 8) and carrying out the integration we find for the numerator of $g(\underline{x})$

$$\sum_{S\subset I}(1-\pi)^s\,\pi^{n-s}\prod_{i\in\bar{S}}p_e(x_i)\int\mu_o(\theta)\prod_{i\in S}p_o(x_{i/\theta})\, dU(\theta)$$

or

$$\sum_{S\subset I}(1-\pi)^s\,\pi^{n-s}\,p_o(x_S)p_e(x_{\bar{S}})\;E_o\left[\mu_o(\theta)/x_S\right]$$

and for the denominator of $g(\underline{x})$

$$\sum_{S\subset I} (1-\pi)^{s} \pi^{n-s} p_o(x_S) p_e(x_{\bar{S}})$$

Dividing both numerator and denominator by $(1-\pi)^n p_o(x_I)$ we finally arrive at

$$10) \quad \boxed{g(\underline{x}) = \frac{E_o\left[\mu_o(\theta)/\underline{x}\right] + \sum\limits_{\substack{S\subset I\\ S\neq I}} L(x_S) E_o\left[\mu_o(\theta)/x_S\right]}{1 + \sum\limits_{\substack{S\subset I\\ S\neq I}} L(x_S)}}$$

<u>**Remarks:**</u>

i) Observe that $g(\underline{x})$ is a weighted average of forecasts based on all subsamples x_S of the total sample x_I, the forecasts being calculated under the assumption that the subsample contains only claims of the ordinary type.

ii) As $\frac{\pi}{1-\pi}$ is usually rather small the weight of $L(x_S)$ is rather quick decreasing with decreasing number of observations in x_S; for a fixed number of observations the weight $L(x_S)$ is rather big if both $p_o(x_S)$ and $p_e(x_{\bar{S}})$ are big i.e. if x_S and $x_{\bar{S}}$ are very likely to come from the ordinary and the excess urn respectively.

iii) Dividing by $p_o(x_I)$ is obviously only allowed if all the observed claims are possibly of ordinary type. The weight function $L(x_S)$ is then only positive if $p_e(x_{\bar{S}})$ is positive i.e. if the subset $x_{\bar{S}}$ is possibly of excess type. Thus the formula does what we would have done by intuition as well, it excludes predictions based on claims which can be surely recognized as excess claims.

5. More insight from the single observation case

At this point it is worthwhile to consider the special case where the whole sample of observations contains only one observation, i.e.

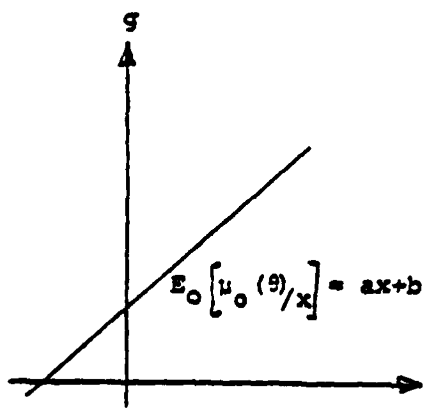
$$\underline{x} = (x_1)$$

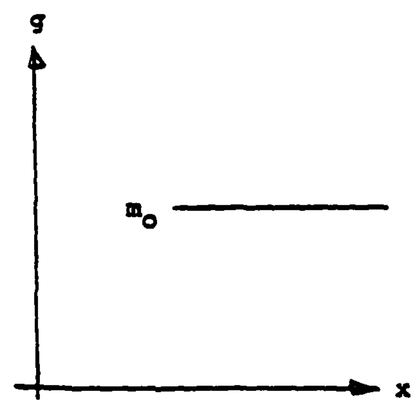
For simplicity we omit the index 1 and write x for the single observation. We have then

$$\text{11)} \qquad g(x) = \frac{E_o\left[\mu_o(\theta)/x\right] + L\left[x_\emptyset\right] E\left[\mu_o(\theta)\right]}{1 + L\left[x_\emptyset\right]}$$

with $$L\left[x_\emptyset\right] = \frac{\pi}{1-\pi}\,\frac{p_e(x)}{p_o(x)}$$

The right hand side is a multiple of the likelihood ratio. If the latter is monotonically increasing (which is typically the case in applications) so is also the weight given to the constant estimator $E\left[\mu_o(\theta)\right] = m_o$. Assume in addition that $E_o\left[\mu_o(\theta)/x\right]$ is of linear form; then our estimator $g(x)$ is a mixture of the two cases (corresponding to the two pictures)





the weight being shifted from the estimator on the left to the estimator on the right as x increases. The resulting estimator is almost of the form a+b min(x,M) . Hence credibility with trimming is almost exact! This fact will be illustrated by a numerical example in section 6. In fact our numerical example will show that this fact also carries over to higher dimensions.

6. A Numerical Example

6.1) For explicit calculations we are assuming that for ordinary claims

$p_o(x_{/\theta})$ is a normal density with mean θ
variance v

θ is normally distributed with mean m_o
variance w

We then have

$p_o(x_S) = \int \prod_{i \in S} p_o(x_{i/\theta})\, dU(\theta)$ which turns out to be a multi-dimensional normal density with mean vector $\begin{pmatrix} m_o \\ m_o \\ \vdots \\ m_o \end{pmatrix}$

and covariance matrix

$$\sum = \begin{pmatrix} w+v & w \ldots w \\ w & w+v \ldots \\ \vdots & \vdots \\ w & \ldots w+v \end{pmatrix}$$

hence

$$12) \quad p_o(x_S) = \frac{\sqrt{|A|}}{(2\pi)^{s/2}} \, e^{-\frac{1}{2} \sum_{\substack{i \in S \\ j \in S}} a_{ij}(x_i - m_o)(x_j - m_o)}$$

with $A = \sum^{-1}$

Proof that $p_o(x_S)$ has density 12:

a) Given θ any linear combination $\sum_{i \in S} c_i x_i$ is normal with mean $\sum_{i \in S} c_i \theta$ and variance $\sum_{i \in S} c_i^2 v$. Integrated out with respect to the normal structure function of θ we obtain a normal distribution with mean $\sum_{i \in S} c_i m_o$ and variance $\left(\sum_{i \in S} c_i\right)^2 w + \sum_{i \in S} c_i^2 v$. But a sample x_S whose linear combinations are all normally distributed is multidimensional normal.

b) Let $\sum = \left(\sigma_{ij}\right)_{\substack{i\in S\\ j\in S}}$

$$\sigma_{ij} = \mathrm{Cov}(X_i,X_j) = E\left[\mathrm{Cov}(X_i,X_j)_{/\theta}\right] + \mathrm{Var}\left[E\left[X_{i/\theta}\right] \cdot E\left[X_{j/\theta}\right]\right]$$

$$= \quad \delta_{ij}\, v \quad + \quad w$$

q.e.d.

It should be noted that

13) $\det \sum = v^n + nv^{n-1}w$ (subtract first row from all other rows and then develop along the first column)

Also observe the explicit form of

$\sum^{-1} = A = \left(a_{ij}\right)_{\substack{i\in S\\ j\in S}}$, namely

14) $a_{ij} = \frac{1}{v}\left(\delta_{ij} - \frac{w}{v+nw}\right)$ $\left[\text{use } (I+\alpha|\underline{\beta})^{-1} = I - \frac{\alpha|\underline{\beta}}{1+\underline{\alpha}\beta|}\right]$

From elementary calculations in credibility theory we finally also know that

15) $E_o\left[\theta/x_S\right] = \bar{x}_S \frac{sw}{v+sw} + m_o \frac{v}{v+sw}$

6.2) For the <u>excess claims</u> the probability law is specified by assuming that

$p_e(x)$ is a normal density with mean μ_e

variance σ_e^2

7. Numerical Calculations of $g(\underline{x})$

For our calculations we have chosen

$$m_o = 10 \qquad \mu_e = 50$$

$$\left.\begin{array}{l} v = 12.5 \\ w = 12.5 \end{array}\right\} \sigma_o = 5 \qquad \sigma_e = 20$$

$$1-\pi = 0.9 \qquad \pi = 0.1$$

and we obtain

a) for n=1 (single observation case)

x	g(x)
5.	7.5091
6.	8.0068
7.	8.5049
8.	9.0033
9.	9.5017
10.	10.0000
11.	10.4979
12.	10.9951
13.	11.4910
14.	11.9850
15.	12.4755
16.	12.9602
17.	13.4348
18.	13.8919
19.	14.31[illegible]7
20.	14.6876
21.	14.9[illegible]
22.	15.[illegible]
23.	14.8[illegible]
24.	14.3712
25.	13.4968
26.	12.4521
27.	11.5065
28.	10.8265
29.	10.4153
30.	10.1952
31.	10.0870
32.	10.0370
33.	10.0151
34.	10.0059
35.	10.0022
36.	10.0008
37.	10.0003
38.	10.0001
39.	10.0000
40.	10.0000

b) for n=2 (two observations) $g(x_1,x_2)$

$x_1 \backslash x_2$	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24
5	6.68	7.02	7.35	7.68	8.02	8.35	8.68	9.01	9.32	9.61	9.86	10.04	10.08	9.92	9.53	8.98	8.43	8.02	7.78	7.65
6	7.02	7.35	7.68	8.01	8.35	8.68	9.01	9.33	9.65	9.94	10.21	10.41	10.50	10.42	10.12	9.62	9.07	8.62	8.32	8.17
7	7.35	7.68	8.01	8.34	8.68	9.01	9.34	9.66	9.98	10.28	10.55	10.78	10.92	10.91	10.70	10.27	9.73	9.24	8.90	8.70
8	7.68	8.01	8.34	8.68	9.01	9.34	9.67	9.99	10.31	10.62	10.90	11.15	11.33	11.38	11.26	10.92	10.42	9.90	9.51	9.26
9	8.02	8.35	8.68	9.01	9.34	9.67	10.00	10.32	10.65	10.96	11.25	11.52	11.72	11.83	11.79	11.55	11.11	10.59	10.14	9.84
10	8.35	8.68	9.01	9.34	9.67	10.00	10.33	10.66	10.98	11.30	11.60	11.88	12.11	12.27	12.30	12.15	11.79	11.30	10.81	10.44
11	8.68	9.01	9.34	9.67	10.00	10.33	10.66	10.99	11.32	11.63	11.94	12.23	12.49	12.69	12.78	12.72	12.46	12.02	11.51	11.08
12	9.01	9.33	9.66	9.99	10.32	10.66	10.99	11.32	11.65	11.97	12.29	12.59	12.86	13.09	13.24	13.26	13.09	12.73	12.23	11.76
13	9.32	9.65	9.98	10.31	10.65	10.98	11.32	11.65	11.98	12.30	12.62	12.93	13.22	13.47	13.67	13.75	13.68	13.41	12.97	12.47
14	9.61	9.94	10.28	10.62	10.96	11.30	11.63	11.97	12.30	12.63	12.96	13.27	13.57	13.85	14.07	14.22	14.23	14.06	13.69	13.20
15	9.86	10.21	10.55	10.90	11.25	11.60	11.94	12.29	12.62	12.96	13.29	13.61	13.92	14.21	14.46	14.65	14.74	14.66	14.39	13.93
16	10.04	10.41	10.78	11.15	11.52	11.88	12.23	12.59	12.93	13.27	13.61	13.94	14.26	14.56	14.83	15.06	15.20	15.21	15.03	14.65
17	10.08	10.50	10.92	11.33	11.72	12.11	12.49	12.86	13.22	13.57	13.92	14.26	14.58	14.90	15.19	15.44	15.62	15.70	15.61	15.31
18	9.92	10.42	10.91	11.38	11.83	12.27	12.69	13.09	13.47	13.85	14.21	14.56	14.90	15.22	15.52	15.79	16.01	16.13	16.12	15.90
19	9.53	10.12	10.70	11.26	11.79	12.30	12.78	13.24	13.67	14.07	14.46	14.83	15.19	15.52	15.84	16.12	16.36	16.51	16.54	16.38
20	8.98	9.62	10.27	10.92	11.55	12.15	12.72	13.26	13.75	14.22	14.65	15.06	15.44	15.79	16.12	16.42	16.66	16.83	16.87	16.73
21	8.43	9.07	9.73	10.42	11.11	11.79	12.46	13.09	13.68	14.23	14.74	15.20	15.62	16.01	16.36	16.66	16.91	17.07	17.10	16.94
22	8.02	8.62	9.24	9.90	10.59	11.30	12.02	12.73	13.41	14.06	14.66	15.21	15.70	16.13	16.51	16.83	17.07	17.21	17.20	16.97
23	7.78	8.32	8.90	9.51	10.14	10.81	11.51	12.23	12.97	13.69	14.39	15.03	15.61	16.12	16.54	16.87	17.10	17.20	17.10	16.76
24	7.65	8.17	8.70	9.26	9.84	10.44	11.08	11.76	12.47	13.20	13.93	14.65	15.31	15.90	16.38	16.73	16.94	16.97	16.76	16.28

c) for n=5 (five observations)

$g(x_1,x_2, C_3,C_4,C_5)$ note: C_3,C_4,C_5 are chosen as "parameters" for the following tables

i) $(C_3,C_4,C_5) = (10,10,10)$

$x_1 \backslash x_2$	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24
5	8.76	8.86	8.98	9.13	9.28	9.43	9.56	9.66	9.71	9.71	9.67	9.60	9.53	9.47	9.42	9.39	9.37	9.36	9.36	9.35
6	8.86	8.94	9.06	9.20	9.35	9.50	9.62	9.71	9.76	9.77	9.73	9.67	9.60	9.54	9.49	9.46	9.44	9.43	9.43	9.43
7	8.98	9.06	9.17	9.31	9.46	9.60	9.72	9.81	9.86	9.87	9.84	9.78	9.71	9.65	9.61	9.58	9.56	9.55	9.54	9.54
8	9.13	9.20	9.31	9.44	9.58	9.72	9.84	9.94	9.99	10.00	9.97	9.92	9.85	9.80	9.75	9.72	9.70	9.69	9.69	9.69
9	9.28	9.35	9.46	9.58	9.72	9.86	9.98	10.08	10.14	10.15	10.13	10.08	10.01	9.96	9.91	9.88	9.86	9.85	9.85	9.84
10	9.43	9.50	9.60	9.72	9.86	10.00	10.12	10.22	10.29	10.31	10.28	10.24	10.17	10.12	10.07	10.04	10.02	10.01	10.00	10.00
11	9.56	9.62	9.72	9.84	9.98	10.12	10.25	10.36	10.42	10.45	10.43	10.38	10.32	10.26	10.22	10.18	10.16	10.15	10.15	10.14
12	9.66	9.71	9.81	9.94	10.08	10.22	10.36	10.46	10.53	10.56	10.55	10.50	10.44	10.38	10.33	10.30	10.28	10.27	10.26	10.26
13	9.72	9.76	9.86	9.99	10.14	10.29	10.42	10.53	10.61	10.64	10.63	10.58	10.52	10.46	10.41	10.37	10.35	10.34	10.33	10.33
14	9.71	9.77	9.87	10.00	10.15	10.31	10.45	10.56	10.64	10.67	10.66	10.61	10.55	10.48	10.43	10.40	10.37	10.36	10.35	10.35
15	9.67	9.73	9.84	9.97	10.13	10.28	10.43	10.55	10.63	10.66	10.64	10.60	10.53	10.47	10.41	10.37	10.35	10.34	10.33	10.33
16	9.60	9.67	9.78	9.92	10.08	10.24	10.38	10.50	10.58	10.61	10.60	10.55	10.48	10.41	10.36	10.32	10.29	10.28	10.27	10.27
17	9.53	9.60	9.71	9.85	10.01	10.17	10.32	10.44	10.52	10.55	10.53	10.48	10.41	10.34	10.29	10.25	10.22	10.21	10.20	10.20
18	9.47	9.54	9.65	9.80	9.96	10.12	10.26	10.38	10.46	10.48	10.47	10.41	10.34	10.27	10.22	10.18	10.16	10.14	10.14	10.13
19	9.42	9.49	9.61	9.75	9.91	10.07	10.22	10.33	10.41	10.43	10.41	10.36	10.29	10.22	10.17	10.13	10.10	10.09	10.09	10.08
20	9.39	9.46	9.58	9.72	9.88	10.04	10.18	10.30	10.37	10.40	10.37	10.32	10.25	10.18	10.13	10.09	10.07	10.06	10.05	10.05
21	9.37	9.44	9.56	9.70	9.86	10.02	10.16	10.28	10.35	10.37	10.35	10.29	10.22	10.16	10.10	10.07	10.05	10.03	10.03	10.02
22	9.36	9.43	9.55	9.69	9.85	10.01	10.15	10.27	10.34	10.36	10.34	10.28	10.21	10.14	10.09	10.06	10.03	10.02	10.02	10.01
23	9.36	9.43	9.54	9.69	9.85	10.00	10.15	10.26	10.33	10.35	10.33	10.27	10.20	10.14	10.09	10.05	10.03	10.02	10.01	10.01
24	9.35	9.43	9.54	9.69	9.84	10.00	10.14	10.26	10.33	10.35	10.33	10.27	10.20	10.13	10.08	10.05	10.02	10.01	10.01	10.00

ii) $(C_3, C_4, C_5) = (10, 10, 25)$

$x_1 \backslash x_2$	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24
5	8.59	8.70	8.84	9.01	9.18	9.35	9.50	9.61	9.66	9.66	9.62	9.54	9.46	9.39	9.33	9.30	9.27	9.26	9.26	9.25
6	8.70	8.80	8.93	9.09	9.26	9.43	9.57	9.67	9.73	9.73	9.69	9.62	9.54	9.47	9.41	9.38	9.36	9.35	9.34	9.34
7	8.84	8.93	9.06	9.22	9.38	9.54	9.68	9.78	9.84	9.85	9.81	9.74	9.67	9.60	9.54	9.51	9.49	9.48	9.47	9.47
8	9.01	9.09	9.22	9.37	9.53	9.68	9.82	9.93	9.99	10.00	9.97	9.90	9.83	9.76	9.71	9.68	9.66	9.65	9.64	9.64
9	9.18	9.26	9.38	9.53	9.69	9.84	9.98	10.09	10.16	10.17	10.15	10.09	10.01	9.95	9.90	9.86	9.84	9.83	9.82	9.82
10	9.35	9.43	9.54	9.68	9.84	10.00	10.14	10.26	10.33	10.35	10.33	10.27	10.20	10.13	10.08	10.05	10.02	10.01	10.01	10.00
11	9.50	9.57	9.68	9.82	9.98	10.14	10.29	10.41	10.48	10.51	10.49	10.44	10.37	10.30	10.25	10.21	10.19	10.18	10.17	10.17
12	9.61	9.67	9.78	9.93	10.09	10.26	10.41	10.53	10.61	10.64	10.63	10.57	10.51	10.44	10.38	10.34	10.32	10.31	10.30	10.30
13	9.66	9.73	9.84	9.99	10.16	10.33	10.48	10.61	10.70	10.73	10.72	10.67	10.60	10.53	10.47	10.43	10.40	10.39	10.38	10.38
14	9.66	9.73	9.85	10.00	10.17	10.35	10.51	10.64	10.73	10.77	10.75	10.70	10.63	10.56	10.50	10.46	10.43	10.42	10.41	10.41
15	9.62	9.69	9.81	9.97	10.15	10.33	10.49	10.63	10.72	10.75	10.74	10.69	10.61	10.54	10.48	10.43	10.41	10.39	10.38	10.38
16	9.54	9.62	9.74	9.90	10.09	10.27	10.44	10.57	10.67	10.70	10.69	10.63	10.55	10.47	10.41	10.37	10.34	10.33	10.32	10.31
17	9.46	9.54	9.67	9.83	10.01	10.20	10.37	10.51	10.60	10.63	10.61	10.55	10.47	10.39	10.33	10.29	10.26	10.24	10.24	10.23
18	9.39	9.47	9.60	9.76	9.95	10.13	10.30	10.44	10.53	10.56	10.54	10.47	10.39	10.32	10.25	10.21	10.18	10.17	10.16	10.16
19	9.33	9.41	9.54	9.71	9.90	10.08	10.25	10.38	10.47	10.50	10.48	10.41	10.33	10.25	10.19	10.15	10.12	10.11	10.10	10.10
20	9.30	9.38	9.51	9.68	9.86	10.05	10.21	10.34	10.43	10.46	10.43	10.37	10.29	10.21	10.15	10.11	10.08	10.07	10.06	10.05
21	9.27	9.36	9.49	9.66	9.84	10.02	10.19	10.32	10.40	10.43	10.41	10.34	10.26	10.18	10.12	10.08	10.05	10.04	10.03	10.03
22	9.26	9.35	9.48	9.65	9.83	10.01	10.18	10.31	10.39	10.42	10.39	10.33	10.24	10.17	10.11	10.07	10.04	10.03	10.02	10.02
23	9.26	9.34	9.47	9.64	9.82	10.01	10.17	10.30	10.38	10.41	10.38	10.32	10.24	10.16	10.10	10.06	10.03	10.02	10.01	10.01
24	9.25	9.34	9.47	9.64	9.82	10.00	10.17	10.30	10.38	10.41	10.38	10.31	10.23	10.16	10.10	10.05	10.03	10.02	10.01	10.01

8. Optimal Trimming

Gisler has shown [1] that for given M the optimal choice of the approximation

$$\widehat{\mu(\theta)} = a + b \sum_{i=1}^{n} (x_i \wedge M) \quad \text{to} \quad \mu(\theta) \quad \left[\text{and hence to } P[\underline{x}]\right]$$

can be calculated as follows

$$16) \quad \tilde{b} = \frac{b_1}{(n-1)b_2 + b_3} \quad \text{where} \quad \begin{aligned} b_1 &= \text{Cov}\,[X_1 \wedge M,\ X_2] \\ b_2 &= \text{Cov}\,[X_1 \wedge M,\ X_2 \wedge M] \\ b_3 &= \text{Var}\,[X_1 \wedge M] \end{aligned}$$

$$17) \quad \tilde{a} + n\tilde{b}\, E[X \wedge M] = E[X]$$

With this optimal choice we then have

$$18) \quad E\left[\left(\widehat{\mu(\theta)} - \mu(\theta)\right)^2\right] = w - n\,\tilde{b} \cdot b_1$$

Hence the trimming point M is optimal if $\tilde{b} \cdot b_1$ is maximum.

In our basic model (cf. section 2) we find

$$19)\quad b_1 = (1-\pi)^2 \operatorname{Cov}\left[\mu_o^M(\theta),\ \mu_o(\theta)\right] \quad \text{where}$$

$$\mu_o^M(\theta) = E\left[X\wedge M/\theta, X \text{ ordinary}\right]$$

$$\mu_o(\theta) = E\left[X/\theta, X \text{ ordinary}\right]$$

$$b_2 = (1-\pi)^2 \operatorname{Var}\left[\mu_o^M(\theta)\right]$$

$$b_3 = (1-\pi) E\left[\sigma_o^{2M}(\theta)\right] + \pi\sigma_e^{2M} + (1-\pi)^2 \operatorname{Var}\left[\mu_o^M(\theta)\right] + $$
$$+\pi(1-\pi)\ E\left[\left(\mu_o(\theta) - \mu_e^M\right)^2\right]$$

$$\text{with } \sigma_o^{2M}(\theta) = \operatorname{Var}\left[X\wedge M/\theta, X \text{ ordinary}\right]$$

$$\sigma_e^{2M} = \operatorname{Var}\left[X\wedge M/X \text{ excess}\right]$$

$$\mu_e^M = E\left[X\wedge M/X \text{ excess}\right]$$

Using explicitely the normal distribution as assumed both for ordinary and excess claims in section 6 we obtain from some rather tedious integrations:

Let $\Phi(.)$ denote the standardized normal distribution function and $\varphi(.)$ the standardized normal density function, then

$$20)\quad b_1 = (1-\pi)^2 w\Phi\left(\frac{M-m_o}{\sigma_o}\right) \qquad \sigma_o = \sqrt{v+w}$$

$$b_2 = (1-\pi)^2 \operatorname{Cov}\left[U_1\wedge M\ ,\ U_2\wedge M\right]$$

where the covariance is obtained by numerical integration.

Notation: (U_1, U_2) is $N\left(\begin{matrix} m_o \\ m_o \end{matrix}\ ,\ \Sigma\right)$ with $\Sigma = \begin{pmatrix} v+w & w \\ w & v+w \end{pmatrix}$

$$b_3 = A - B^2$$

$$A = (1-\pi)\left[(m_o^2+\sigma_o^2)\ \Phi\left(\frac{M-m_o}{\sigma_o}\right) - \sigma_o(M+m\ )\ \varphi\left(\frac{M-m_o}{\sigma_o}\right)\right]$$
$$+\ \pi\ \left[(\mu_e^2+\sigma_e^2)\ \Phi\left(\frac{M-\mu_e}{\sigma_e}\right) - \sigma_e(M+\mu_e)\ \varphi\left(\frac{M-\mu_e}{\sigma_e}\right)\right]$$
$$+\ M^2\ \left[1-(1-\pi)\Phi\left(\frac{M-m_o}{\sigma_o}\right) - \pi\Phi\left(\frac{M-\mu_e}{\sigma_e}\right)\right]$$

$$B = (1-\pi)\ \left[(m_o-M)\ \Phi\left(\frac{M-m_o}{\sigma_o}\right) - \sigma_o\varphi\left(\frac{M-m_o}{\sigma_o}\right)\right]$$
$$+\ \pi\ \left[(\mu_e-M)\ \Phi\left(\frac{M-\mu_e}{\sigma_e}\right) - \sigma_e\varphi\left(\frac{M-\mu_e}{\sigma_e}\right)\right]$$
$$+\ M$$

9. Numerical Calculations of $\tilde{a} + \tilde{b} \sum_{i=1}^{n} (x_i \wedge \tilde{M})$

Using the same parameter values as in section 7 we obtain the forecasts based on optimal trimming. To compare with $g(\underline{x})$ it is worthwhile to calculate also $\tilde{a}' + \tilde{b}' \sum_{i=1}^{n} (x_i \wedge \tilde{M})$ with

$$\tilde{a}' = \frac{\tilde{a} - \pi\mu_e}{1-\pi} \qquad\qquad \tilde{b}' = \frac{\tilde{b}}{1-\pi}$$

a) Results for n=1 (single observation case)

Truncation point $\tilde{M} = 14.68$

formula: $\hat{P}=0.4412(x\wedge\tilde{M})+9.5817$ $\hat{g}=0.4902(x\wedge\tilde{M})+5.0908$

x	$\hat{P}$	$\hat{g}$
5	11.79	7.54
6	12.23	8.03
7	12.67	8.52
8	13.11	9.01
9	13.55	9.50
10	13.99	9.99
11	14.43	10.48
12	14.88	10.97
13	15.32	11.46
14	15.76	11.95
15	16.06	12.29
16	16.06	12.29
17	16.06	12.29
18	16.06	12.29
19	16.06	12.29
20	16.06	12.29

b) Results for n=2

truncation point $\tilde{M} = 19.52$

b_1) approximation to total premium $P[\underline{x}]$

formula: $\hat{P} = 0.2289 \sum_{i=1}^{2} (x_i \wedge \tilde{M}) + 9.0351$

$x_1 \backslash x_2$	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24
5	11.32	11.55	11.78	12.01	12.24	12.47	12.70	12.93	13.16	13.38	13.61	13.84	14.07	14.30	14.53	14.65	14.65	14.65	14.65	14.65
6	11.55	11.78	12.01	12.24	12.47	12.70	12.93	13.16	13.38	13.61	13.84	14.07	14.30	14.53	14.76	14.88	14.88	14.88	14.88	14.88
7	11.78	12.01	12.24	12.47	12.70	12.93	13.16	13.38	13.61	13.84	14.07	14.30	14.53	14.76	14.99	15.11	15.11	15.11	15.11	15.11
8	12.01	12.24	12.47	12.70	12.93	13.16	13.38	13.61	13.84	14.07	14.30	14.53	14.76	14.99	15.22	15.33	15.33	15.33	15.33	15.33
9	12.24	12.47	12.70	12.93	13.16	13.38	13.61	13.84	14.07	14.30	14.53	14.76	14.99	15.22	15.44	15.56	15.56	15.56	15.56	15.56
10	12.47	12.70	12.93	13.16	13.38	13.61	13.84	14.07	14.30	14.53	14.76	14.99	15.22	15.44	15.67	15.79	15.79	15.79	15.79	15.79
11	12.70	12.93	13.26	13.38	13.61	13.84	14.07	14.30	14.53	14.76	14.99	15.22	15.44	15.67	15.90	16.02	16.02	16.02	16.02	16.02
12	12.93	13.16	13.38	13.61	13.84	14.07	14.30	14.53	14.76	14.99	15.22	15.44	15.67	15.90	16.13	16.25	16.25	16.25	16.25	16.25
13	13.16	13.38	13.61	13.84	14.07	14.30	14.53	14.76	14.99	15.22	15.44	15.67	15.90	16.13	16.36	16.48	16.48	16.48	16.48	16.48
14	13.38	13.61	13.84	14.07	14.30	14.53	14.76	14.99	15.22	15.44	15.67	15.90	16.13	16.36	16.59	16.71	16.71	16.71	16.71	16.71
15	13.61	13.84	14.07	14.30	14.53	14.76	14.99	15.22	15.44	15.67	15.90	16.13	16.36	16.59	16.82	16.94	16.94	16.94	16.94	16.94
16	13.84	14.07	14.30	14.53	14.76	14.99	15.22	15.44	15.67	15.90	16.13	16.36	16.59	16.82	17.05	17.17	17.17	17.17	17.17	17.17
17	14.07	14.30	14.53	14.76	14.99	15.22	15.44	15.67	15.90	16.13	16.36	16.59	16.82	17.05	17.28	17.39	17.39	17.39	17.39	17.39
18	14.30	14.53	14.76	14.99	15.22	15.44	15.67	15.90	16.13	16.36	16.59	16.82	17.05	17.28	17.50	17.62	17.62	17.62	17.62	17.62
19	14.53	14.76	14.99	15.22	15.44	15.67	15.90	16.13	16.36	16.59	16.82	17.05	17.28	17.50	17.73	17.85	17.85	17.85	17.85	17.85
20	14.65	14.88	15.11	15.33	15.56	15.79	16.02	16.25	16.48	16.71	16.94	17.17	17.39	17.62	17.85	17.97	17.97	17.97	17.97	17.97
21	14.65	14.88	15.11	15.33	15.56	15.79	16.02	16.25	16.48	16.71	16.94	17.17	17.39	17.62	17.85	17.97	17.97	17.97	17.97	17.97
22	14.65	14.88	15.11	15.33	15.56	15.79	16.02	16.25	16.48	16.71	16.94	17.17	17.39	17.62	17.85	17.97	17.97	17.97	17.97	17.97
23	14.65	14.88	15.11	15.33	15.56	15.79	16.02	16.25	16.48	16.71	16.94	17.17	17.39	17.62	17.85	17.97	17.97	17.97	17.97	17.97
24	14.65	14.88	15.11	15.33	15.56	15.79	16.02	16.25	16.48	16.71	16.94	17.17	17.39	17.62	17.85	17.97	17.97	17.97	17.97	17.97

b_2) approximation to ordinary premium $g(\underline{x})$

formula: $\hat{g} = 0.2543 \sum_{i=1}^{2} (x_i \wedge \tilde{M}) + 4.4834$

$x_1 \backslash x_2$	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24
5	7.03	7.28	7.53	7.79	8.04	8.30	8.55	8.81	9.06	9.32	9.57	9.82	10.08	10.33	10.59	10.72	10.72	10.72	10.72	10.72
6	7.28	7.53	7.79	8.04	8.30	8.55	8.81	9.06	9.32	9.57	9.82	10.08	10.33	10.59	10.84	10.97	10.97	10.97	10.97	10.97
7	7.53	7.79	8.04	8.30	8.55	8.81	9.06	9.32	9.57	9.82	10.08	10.33	10.59	10.84	11.10	11.23	11.23	11.23	11.23	11.23
8	7.79	8.04	8.30	8.55	8.81	9.06	9.32	9.57	9.82	10.08	10.33	10.59	10.84	11.10	11.35	11.48	11.48	11.48	11.48	11.48
9	8.04	8.30	8.55	8.81	9.06	9.32	9.57	9.82	10.08	10.33	10.59	10.84	11.10	11.35	11.60	11.74	11.74	11.74	11.74	11.74
10	8.30	8.55	8.81	9.06	9.32	9.57	9.82	10.08	10.33	10.59	10.84	11.10	11.35	11.60	11.86	11.99	11.79	11.99	11.99	11.99
11	8.55	8.81	9.06	9.32	9.57	9.82	10.08	10.33	10.59	10.84	11.10	11.35	11.60	11.86	12.11	12.24	12.24	12.24	12.24	12.24
12	8.81	9.06	9.32	9.57	9.82	10.08	10.33	10.59	10.84	11.10	11.35	11.60	11.86	12.11	12.37	12.50	12.50	12.50	12.50	12.50
13	9.06	9.32	9.57	9.82	10.08	10.33	10.59	10.84	11.10	11.35	11.60	11.86	12.11	12.37	12.62	12.75	12.75	12.75	12.75	12.75
14	9.32	9.57	9.82	10.08	10.33	10.59	10.84	11.10	11.35	11.60	11.86	12.11	12.37	12.62	12.88	13.01	13.01	13.01	13.01	13.01
15	9.57	9.82	10.08	10.33	10.59	10.84	11.10	11.35	11.60	11.86	12.11	12.37	12.62	12.88	13.13	13.26	13.26	13.26	13.26	13.26
16	9.82	10.08	10.33	10.59	10.84	11.10	11.35	11.60	11.86	12.11	12.37	12.62	12.88	13.13	13.38	13.52	13.52	13.52	13.52	13.52
17	10.08	10.33	10.59	10.84	11.10	11.35	11.60	11.86	12.11	12.37	12.62	12.88	13.13	13.38	13.64	13.77	13.77	13.77	13.77	13.77
18	10.33	10.59	10.84	11.10	11.35	11.60	11.86	12.11	12.37	12.62	12.38	13.13	13.38	13.64	13.89	14.02	14.02	14.02	14.02	14.02
19	10.59	10.84	11.10	11.35	11.60	11.86	12.11	12.37	12.62	12.88	13.13	13.38	13.64	13.89	14.15	14.28	14.28	14.28	14.28	14.28
20	10.72	10.97	11.23	11.48	11.74	11.99	12.24	12.50	12.75	13.01	13.26	13.52	13.77	14.02	14.28	14.41	14.41	14.41	14.41	14.41
21	10.72	10.97	11.23	11.48	11.74	11.99	12.24	12.50	12.75	13.01	13.26	13.52	13.77	14.02	14.28	14.41	14.41	14.41	14.41	14.41
22	10.72	10.97	11.23	11.48	11.74	11.99	12.24	12.50	12.75	13.01	13.26	13.52	13.77	14.02	14.28	14.41	14.41	14.41	14.41	14.41
23	10.72	10.97	11.23	11.48	11.74	11.99	12.24	12.50	12.75	13.01	13.26	13.52	13.77	14.02	14.28	14.41	14.41	14.41	14.41	14.41
24	10.72	10.97	11.23	11.48	11.74	11.99	12.24	12.50	12.75	13.01	13.26	13.52	13.77	14.02	14.28	14.41	14.41	14.41	14.41	14.41

c) Results for n=5

truncation point $\tilde{M} = 22.83$

formulae: $\hat{P} = 0.1241 \sum_{i=1}^{5} (x_i \wedge \tilde{M}) + 7.0561$ total premium

$\hat{g} = 0.1378 \sum_{i=1}^{5} (x_i \wedge \tilde{M}) + 2.2845$ ordinary premium

c_1) approximation to total premium $P[\underline{x}]$ $= P[x_1, x_2, \underbrace{C_3, C_4, C_5}_{\text{chosen as fixed parameter values}}]$

i) $(C_3, C_4, C_5) = (10, 10, 10)$

$x_1 \backslash x_2$	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24
5	12.02	12.14	12.27	12.39	12.52	12.64	12.76	12.89	13.01	13.14	13.26	13.39	13.51	13.63	13.76	13.88	14.01	14.13	14.23	14.23
6	12.14	12.27	12.39	12.52	12.64	12.76	12.89	13.01	13.14	13.26	13.39	13.51	13.63	13.76	13.88	14.01	14.13	14.25	14.36	14.36
7	12.27	12.39	12.52	12.64	12.76	12.89	13.01	13.14	13.26	13.39	13.51	13.63	13.76	13.88	14.01	14.13	14.25	14.38	14.48	14.48
8	12.39	12.52	12.64	12.76	12.89	13.01	13.14	13.26	13.39	13.51	13.63	13.76	13.88	14.01	14.13	14.25	14.38	14.50	14.61	14.61
9	12.52	12.64	12.76	12.89	13.01	13.14	13.26	13.39	13.51	13.63	13.76	13.88	14.01	14.13	14.25	14.38	14.50	14.63	14.73	14.73
10	12.64	12.76	12.89	13.01	13.14	13.26	13.39	13.51	13.63	13.76	13.88	14.01	14.13	14.25	14.38	14.50	14.63	14.75	14.85	14.85
11	12.76	12.89	13.01	13.14	13.26	13.39	13.51	13.63	13.76	13.88	14.01	14.13	14.25	14.38	14.50	14.63	14.75	14.87	14.98	14.98
12	12.89	13.01	13.14	13.26	13.39	13.51	13.63	13.76	13.88	14.01	14.13	14.25	14.38	14.50	14.63	14.75	14.87	15.00	15.10	15.10
13	13.01	13.14	13.26	13.39	13.51	13.63	13.76	13.88	14.01	14.13	14.25	14.38	14.50	14.63	14.75	14.87	15.00	15.12	15.23	15.23
14	13.14	13.26	13.39	13.51	13.63	13.76	13.88	14.01	14.13	14.25	14.38	14.50	14.63	14.75	14.87	15.00	15.12	15.25	15.35	15.35
15	13.26	13.39	13.51	13.63	13.76	13.88	14.01	14.13	14.25	14.38	14.50	14.63	14.75	14.87	15.00	15.12	15.25	15.37	15.47	15.47
16	13.39	13.51	13.63	13.76	13.88	14.01	14.13	14.25	14.38	14.50	14.63	14.75	14.87	15.00	15.12	15.25	15.37	15.49	15.60	15.60
17	13.51	13.63	13.76	13.88	14.01	14.13	14.25	14.38	14.50	14.63	14.75	14.87	15.00	15.12	15.25	15.37	15.49	15.62	15.72	15.72
18	13.63	13.76	13.88	14.01	14.13	14.25	14.38	14.50	14.62	14.75	14.87	15.00	15.12	15.25	15.37	15.49	15.62	15.74	15.85	15.85
19	13.76	13.88	14.01	14.13	14.25	14.38	14.50	14.63	14.75	14.87	15.00	15.12	15.25	15.37	15.49	15.62	15.74	15.87	15.97	15.97
20	13.88	14.01	14.13	14.25	14.38	14.50	14.63	14.75	14.87	15.00	15.12	15.25	15.37	15.49	15.62	15.74	15.87	15.99	16.09	16.09
21	14.01	14.13	14.25	14.38	14.50	14.63	14.75	14.87	15.00	15.12	15.25	15.37	15.49	15.62	15.74	15.87	15.99	16.12	16.22	16.22
22	14.13	14.25	14.38	14.50	14.63	14.75	14.87	15.00	15.12	15.25	15.37	15.49	15.62	15.74	15.87	15.99	16.12	16.24	16.34	16.34
23	14.23	14.36	14.48	14.61	14.73	14.85	14.98	15.10	15.23	15.35	15.47	15.60	15.72	15.85	15.97	16.09	16.22	16.34	16.45	16.45
24	14.23	14.36	14.48	14.61	14.73	14.85	14.98	15.10	15.23	15.35	15.47	15.60	15.72	15.85	15.97	16.09	16.22	16.34	16.45	16.45

ii) $(C_3,C_4,C_5) = (10,10,25)$

$x_1 \backslash x_2$	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24
5	13.61	13.74	13.86	13.98	14.11	14.23	14.36	14.48	14.61	14.73	14.85	14.98	15.10	15.23	15.35	15.47	15.60	15.72	15.83	15.83
6	13.74	13.86	13.98	14.11	14.23	14.36	14.48	14.61	14.73	14.85	14.98	15.10	15.23	15.35	15.47	15.60	15.72	15.85	15.95	15.95
7	13.86	13.98	14.11	14.23	14.36	14.48	14.61	14.73	14.85	14.98	15.10	15.23	15.35	15.47	15.60	15.72	15.85	15.97	16.07	16.07
8	13.98	14.11	14.23	14.36	14.48	14.61	14.73	14.85	14.98	15.10	15.23	15.35	15.47	15.60	15.72	15.85	15.97	16.09	16.20	16.20
9	14.11	14.23	14.36	14.48	14.61	14.73	14.85	14.98	15.10	15.23	15.35	15.47	15.60	15.72	15.85	15.97	16.09	16.22	16.32	16.32
10	14.23	14.36	14.48	14.61	14.73	14.85	14.98	15.10	15.23	15.35	15.47	15.60	15.72	15.85	15.97	16.09	16.22	16.34	16.45	16.45
11	14.36	14.48	14.61	14.73	14.85	14.98	15.10	15.23	15.35	15.47	15.60	15.72	15.85	15.97	16.09	16.22	16.34	16.47	16.57	16.57
12	14.48	14.61	14.73	14.85	14.98	15.10	15.23	15.35	15.47	15.60	15.72	15.85	15.97	16.09	16.22	16.34	16.47	16.59	16.69	16.69
13	14.61	14.73	14.85	14.98	15.10	15.23	15.35	15.47	15.60	15.72	15.85	15.97	16.09	16.22	16.34	16.47	16.59	16.71	16.82	16.82
14	14.73	14.85	14.98	15.10	15.23	15.35	15.47	15.60	15.72	15.85	15.97	16.09	16.22	16.34	16.47	16.59	16.71	16.84	16.94	16.94
15	14.85	14.98	15.10	15.23	15.35	15.47	15.60	15.72	15.85	15.97	16.09	16.22	16.34	16.47	16.59	16.71	16.84	16.96	17.07	17.07
16	14.98	15.10	15.23	15.35	15.47	15.60	15.72	15.85	15.97	16.09	16.22	16.34	16.47	16.59	16.71	16.84	16.96	17.09	17.19	17.19
17	15.10	15.23	15.35	[illegible]	15.60	15.72	15.85	15.97	16.09	16.22	16.34	16.47	16.59	16.71	16.84	16.96	17.09	17.21	17.31	17.31
18	15.23	15.35	15.47	15.60	15.72	15.85	15.97	16.09	16.22	16.34	16.47	16.59	16.71	16.84	16.96	17.09	17.21	17.34	17.44	17.44
19	15.35	15.47	15.60	15.72	15.85	15.97	16.09	16.22	16.34	16.47	16.59	16.71	16.84	16.96	17.09	17.21	17.34	17.46	17.56	17.56
20	15.47	15.60	15.72	15.85	15.97	16.09	16.22	16.34	16.47	16.59	16.71	16.84	16.96	17.09	17.21	17.34	17.46	17.58	17.69	17.69
21	15.60	15.72	15.85	15.97	16.09	16.22	16.34	16.47	16.59	16.71	16.84	16.96	17.09	17.21	17.34	17.46	17.58	17.71	17.81	17.81
22	15.72	15.85	15.97	16.09	16.22	16.34	16.47	16.59	16.71	16.84	16.96	17.09	17.21	17.34	17.46	17.58	17.71	17.83	17.93	17.93
23	15.83	15.95	16.07	16.20	16.32	16.45	16.57	16.69	16.82	16.94	17.07	17.19	17.31	17.44	17.56	17.69	17.81	17.93	18.04	18.04
24	15.83	15.95	16.07	16.20	16.32	16.45	16.57	16.69	16.82	16.94	17.07	17.19	17.31	17.44	17.56	17.69	17.81	17.93	18.04	18.04

$\underline{c_2}$) <u>approximation to ordinary premium</u> $g(\underline{x}) = g(x_1, x_2, \underbrace{C_1, C_2, C_3}_{\text{chosen as fixed parameters}})$

i) $(C_3,C_4,C_5) = (10,10,10)$

$x_1 \backslash x_2$	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24
5	7.80	7.93	8.07	8.21	8.35	8.49	8.62	8.76	8.90	9.04	9.17	9.31	9.45	9.59	9.73	9.86	10.00	10.14	10.25	10.25
6	7.93	8.07	8.21	8.35	8.49	8.62	8.76	8.90	9.04	9.17	9.31	9.45	9.59	9.73	9.86	10.00	10.14	10.28	10.39	10.39
7	8.07	8.21	8.35	8.49	8.62	8.76	8.90	9.04	9.17	9.31	9.45	9.59	9.73	9.86	10.00	10.14	10.28	10.41	10.53	10.53
8	8.21	8.35	8.49	8.62	8.76	8.90	9.04	9.17	9.31	9.45	9.59	9.73	9.86	10.00	10.14	10.28	10.41	10.55	10.67	10.67
9	8.35	8.49	8.62	8.76	8.90	9.04	9.17	9.31	9.45	9.59	9.73	9.86	10.00	10.14	10.28	10.41	10.55	10.69	10.80	10.80
10	8.49	8.62	8.76	8.90	9.04	9.17	9.31	9.45	9.59	9.73	9.86	10.00	10.14	10.28	10.41	10.55	10.69	10.83	10.94	10.94
11	8.62	8.76	8.90	9.04	9.17	9.31	9.45	9.59	9.73	9.86	10.00	10.14	10.28	10.41	10.55	10.69	10.83	10.97	11.08	11.08
12	8.76	8.90	9.04	9.17	9.31	9.45	9.59	9.73	9.86	10.00	10.14	10.28	10.41	10.55	10.69	10.83	10.97	11.10	11.22	11.22
13	8.90	9.04	9.17	9.31	9.45	9.59	9.73	9.86	10.00	10.14	10.28	10.41	10.55	10.69	10.83	10.97	11.10	11.24	11.36	11.36
14	9.04	9.17	9.31	9.45	9.59	9.73	9.86	10.00	10.14	10.28	10.41	10.55	10.69	10.83	10.97	11.10	11.24	11.38	11.49	11.49
15	9.17	9.31	9.45	9.59	9.73	9.86	10.00	10.14	10.28	10.41	10.55	10.69	10.83	10.97	11.10	11.24	11.38	11.52	11.63	11.63
16	9.31	9.45	9.59	9.73	9.86	10.00	10.14	10.28	10.41	10.55	10.69	10.83	10.97	11.10	11.24	11.38	11.52	11.65	11.77	11.77
17	9.45	9.59	9.73	9.86	10.00	10.14	10.28	10.41	10.55	10.69	10.83	10.97	11.10	11.24	11.38	11.52	11.65	11.79	11.91	11.91
18	9.59	9.73	9.86	10.00	10.14	10.28	10.41	10.55	10.69	10.83	10.97	11.10	11.24	11.38	11.52	11.65	11.79	11.93	12.04	12.04
19	9.73	9.86	10.00	10.14	10.28	10.41	10.55	10.69	10.83	10.97	11.10	11.24	11.38	11.52	11.65	11.79	11.93	12.07	12.18	12.18
20	9.86	10.00	10.14	10.28	10.41	10.55	10.69	10.83	10.97	11.10	11.24	11.38	11.52	11.65	11.79	11.93	12.07	12.21	12.32	12.32
21	10.00	10.14	10.28	10.41	10.55	10.69	10.83	10.97	11.10	11.24	11.38	11.52	11.65	11.79	11.93	12.07	12.21	12.34	12.46	12.46
22	10.14	10.28	10.41	10.55	10.69	10.83	10.97	11.10	11.24	11.38	11.52	11.65	11.79	11.93	12.07	12.21	12.34	12.48	12.60	12.60
23	10.25	10.39	10.53	10.67	10.80	10.94	11.08	11.22	11.36	11.49	11.63	11.77	11.91	12.04	12.18	12.32	12.46	12.60	12.71	12.71
24	10.25	10.39	10.53	10.67	10.80	10.94	11.08	11.22	11.36	11.49	11.63	11.77	11.31	12.04	12.18	12.32	12.46	12.60	12.71	12.71

ii) $(C_3, C_4, C_5) = (10,10,25)$

$z_1 \backslash z_2$	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24
5	9.56	9.70	9.84	9.98	10.12	10.25	10.39	10.53	10.67	10.80	10.94	11.08	11.22	11.36	11.49	11.63	11.77	11.91	12.02	12.02
6	9.70	9.84	9.98	10.12	10.25	10.39	10.53	10.67	10.80	10.94	11.08	11.22	11.36	11.49	11.63	11.77	11.91	12.04	12.16	12.16
7	9.84	9.98	10.12	10.25	10.39	10.53	10.67	10.80	10.94	11.08	11.22	11.36	11.49	11.63	11.77	11.91	12.04	12.18	12.30	12.30
8	9.98	10.12	10.25	10.39	10.53	10.67	10.80	10.94	11.08	11.22	11.36	11.49	11.63	11.77	11.91	12.04	12.18	12.32	12.43	12.43
9	10.12	10.25	10.39	10.53	10.67	10.80	10.94	11.08	11.22	11.36	11.49	11.63	11.77	11.91	12.04	12.18	12.32	12.46	12.57	12.57
10	10.25	10.39	10.53	10.67	10.80	10.94	11.08	11.22	11.36	11.49	11.63	11.77	11.91	12.04	12.18	12.32	12.46	12.60	12.71	12.71
11	10.39	10.53	10.67	10.80	10.94	11.08	11.22	11.36	11.49	11.63	11.77	11.91	12.04	12.18	12.32	12.46	12.60	12.73	12.85	12.85
12	10.53	10.67	10.80	10.94	11.08	11.22	11.36	11.49	11.63	11.77	11.91	12.04	12.18	12.32	12.46	12.60	12.73	12.87	12.99	12.99
13	10.67	10.80	10.94	11.08	11.22	11.36	11.49	11.63	11.77	11.91	12.04	12.18	12.32	12.46	12.60	12.73	12.87	13.01	13.12	13.12
14	10.80	10.94	11.08	11.22	11.36	11.49	11.63	11.77	11.91	12.04	12.18	12.32	12.46	12.60	12.73	12.87	13.01	13.15	13.26	13.26
15	10.94	11.08	11.22	11.36	11.49	11.63	11.77	11.91	12.04	12.18	12.32	12.46	12.60	12.73	12.87	13.01	13.15	13.29	13.40	13.40
16	11.08	11.22	11.36	11.49	11.63	11.77	11.91	12.04	12.18	12.32	12.46	12.60	12.73	12.87	13.01	13.15	13.29	13.42	13.54	13.54
17	11.22	11.36	11.49	11.63	11.77	11.91	12.04	12.18	12.32	12.46	12.60	12.73	12.87	13.01	13.15	13.29	13.42	13.55	13.68	13.68
18	11.36	11.49	11.63	11.77	11.91	12.04	12.18	12.32	12.46	12.60	12.73	12.87	13.01	13.15	13.29	13.42	13.56	13.70	13.81	13.81
19	11.49	11.63	11.77	11.91	12.04	12.18	12.32	12.46	12.60	12.73	12.87	13.01	13.15	13.29	13.42	13.56	13.70	13.84	13.95	13.95
20	11.63	11.77	11.91	12.04	12.18	12.32	12.46	12.60	12.73	12.87	13.01	13.15	13.29	13.42	13.56	13.70	13.84	13.97	14.09	14.09
21	11.77	11.91	12.04	12.18	12.32	12.46	12.60	12.73	12.87	13.01	13.15	13.29	13.42	13.56	13.70	13.84	13.97	14.11	14.23	14.23
22	11.91	12.04	12.18	12.32	12.46	12.60	12.73	12.87	13.01	13.15	13.29	13.42	13.56	13.70	13.84	13.97	14.11	14.25	14.36	14.36
23	12.02	12.16	12.30	12.43	12.57	12.71	12.85	12.99	13.12	13.26	13.40	13.54	13.68	13.81	13.95	14.09	14.23	14.36	14.48	14.48
24	12.02	12.16	12.30	12.43	12.57	12.71	12.85	12.99	13.12	13.26	13.40	13.54	13.68	13.81	13.95	14.09	14.23	14.36	14.48	14.48

10. Final Remarks

The Data Trimmed Credibility Formulae seem quite appropriate for Experience Rating in the presence of catastrophic (or as called in this paper excess) claims. With this intuitive background in our minds we have in our explicit calculations been looking at deviations from ordinary claims towards the higher side only. Obviously the normal distribution being symmetric one could also observe "outliers" to ordinary claims towards the lower side hence leading to a truncation at the lower end as well. But of course our assumption of normally distributed claims should only be seen as an approximation to the real world, and it is our feeling that the approximation is particularly bad at the lower tail of the distribution.

In any case truncation at the upper end of the distribution is introducing an additional parameter into the credibility formulae and we hope to have demonstrated in this paper that the labour caused by the new parameter can be worthwhile indeed.

11. Bibliography

[1] A. Gisler Optimales Stutzen von Beobachtungen im Credibility Modell (ETH Thesis 1980)

see also A. Gisler Optimum Trimming of Data in the Credibility Model BASA 1980 (3)

12. Appendix

For the interested reader we are attaching the explicit calculations leading to formulae 19) and 20).

A: Calculations leading to formula 19)

$$b_1 = E\left[\mathrm{Cov}[X_1 \wedge M,\ X_{2/\theta}]\right] + \mathrm{Cov}\left[E[X_1 \wedge M_{/\theta}],\ E[X_{2/\theta}]\right]$$

$\mathrm{Cov}[X_1 \wedge M,\ X_{2/\theta}] = 0$, because X_1, X_2 are conditionally independent.

Hence

$$b_1 = \mathrm{Cov}[(1-\pi)\mu_o^M(\theta) + \pi\mu_e^M\ ,\ (1-\pi)\mu_o(\theta) + \pi\mu_e]\ ,\quad \text{or}$$

$$\boxed{b_1 = (1-\pi)^2\ \mathrm{Cov}[\mu_o^M(\theta),\ \mu_o(\theta)]}$$

and analogously (with $X_2 \wedge M$ instead of X_2)

$$\boxed{b_2 = (1-\pi)^2\ \mathrm{Var}\ [\mu_o^M(\theta)]}\ .$$

Let be $Y = 1_A$ where A denotes the event {X is ordinary}. Then

$$\mathrm{Var}[X \wedge M_{/\theta}] = E\left[\mathrm{Var}[X \wedge M_{/\theta,Y}]\right] + \mathrm{Var}\left[E[X \wedge M_{/\theta,Y}]\right]$$

$$= (1-\pi)\sigma_o^{2M}(\theta) + \pi\sigma_e^{2M} + \pi(1-\pi)(\mu_o^M(\theta) - \mu_e^M)^2$$

Hence

$$b_3 = \mathrm{Var}[X \wedge M]$$

$$= E\left[\mathrm{Var}[X \wedge M_{/\theta}]\right] + \mathrm{Var}[(1-\pi)\mu_o^M(\theta) + \mu_e^M]\ ,\ \text{or}$$

$$\boxed{b_3 = (1-\pi)\ E[\sigma_o^{2M}(\theta)] + \pi\sigma_e^{2M} + \pi(1-\pi)\ E[(\mu_o^M(\theta) - \mu_e^M)^2] + (1-\pi)^2 \mathrm{Var}\ \mu_o^M(\theta)]}$$

B: Calculations leading to formula 20)

i) Preparations

In the following we put $r=\sqrt{v}$, $s=\sqrt{w}$ and $\sigma_0=\sqrt{v+w}=\sqrt{r^2+s^2}$. Furthermore we denote by $\Phi(x)$ the standardized normal distribution function and by $\varphi(x)$ the standardized normal density function.

By convolution we get

$$\int_{-\infty}^{\infty} \frac{1}{s}\varphi\left(\frac{x-\mu}{s}\right)\Phi\left(\frac{M-x}{r}\right)dx = \Phi\left(\frac{M-\mu}{\sigma_0}\right)$$

$$\int_{-\infty}^{\infty} \frac{1}{rs}\varphi\left(\frac{x-\mu}{s}\right)\varphi\left(\frac{M-x}{r}\right)dx = \frac{1}{\sigma_0}\varphi\left(\frac{M-\mu}{\sigma_0}\right)$$

Noting that $\varphi'(x) = -x\,\varphi(x)$ integration by parts gives

$$\int_{-\infty}^{\infty} (x-\mu)\varphi\left(\frac{x-\mu}{s}\right)\Phi\left(\frac{M-x}{r}\right)dx = -\frac{s^2}{r}\int_{-\infty}^{\infty}\varphi\left(\frac{x-\mu}{s}\right)\varphi\left(\frac{M-x}{r}\right)dx$$

and thus

$$\int_{-\infty}^{\infty} x\varphi\left(\frac{x-\mu}{s}\right)\Phi\left(\frac{M-x}{r}\right)dx = \mu s\Phi\left(\frac{M-\mu}{\sigma_0}\right)-\frac{s^3}{\sigma_0}\varphi\left(\frac{M-\mu}{\sigma_0}\right)$$

Because of $\varphi\left(\frac{x-\mu}{s}\right)\varphi\left(\frac{M-x}{r}\right) = \varphi\left(\frac{M-\mu}{\sigma_0}\right)\varphi\left(\frac{x-\tilde{\mu}}{\tilde{\sigma}}\right)$

where $\tilde{\mu} = \dfrac{r^2\mu+s^2M}{r^2+s^2}$ and $\tilde{\sigma} = \dfrac{rs}{\sigma_0}$

we obtain

$$\int_{-\infty}^{\infty} x\varphi\left(\frac{x-\mu}{s}\right)\varphi\left(\frac{M-x}{r}\right)dx = \varphi\left(\frac{M-\mu}{\sigma_0}\right)\cdot\tilde{\mu}\cdot\tilde{\sigma}$$

$$\int_{-\infty}^{\infty} x^2\varphi\left(\frac{x-\mu}{s}\right)\varphi\left(\frac{M-x}{r}\right)dx = \varphi\left(\frac{M-\mu}{\sigma_0}\right)\cdot\tilde{\sigma}\cdot(\tilde{\mu}^2+\tilde{\sigma}^2)$$

Integration by parts gives

$$\int_{-\infty}^{\infty} x(x-\mu)\ \varphi\left(\frac{x-\mu}{s}\right)\Phi\left(\frac{M-x}{r}\right)dx = s^2\int_{-\infty}^{\infty}\varphi\left(\frac{x-\mu}{s}\right)\Phi\left(\frac{M-x}{r}\right)dx$$

$$-\frac{s^2}{r}\int_{-\infty}^{\infty} x\varphi\left(\frac{x-\mu}{s}\right)\varphi\left(\frac{M-x}{r}\right)dx$$

and thus using the above formulae

$$\int_{-\infty}^{\infty} x^2\varphi\left(\frac{x-\mu}{s}\right)\Phi\left(\frac{M-x}{r}\right)dx = s(\mu^2+s^2)\Phi\left(\frac{M-\mu}{\sigma_o}\right)-\left(\frac{s}{\sigma_o}\right)^3(2r^2\mu+(M+\mu)s^2)\varphi\left(\frac{M-\mu}{\sigma_o}\right)$$

ii) actual calculations

$$\mu_o^M(\theta) = \int_{-\infty}^{M}\frac{x}{r}\varphi\left(\frac{x-\theta}{r}\right)dx + M\cdot\Pr[X \geq M_{/\theta}]$$

$$= -r\varphi\left(\frac{M-\theta}{r}\right) + \theta\ \Pr[X \leq M_{/\theta}] + M\ \Pr[X \geq M_{/\theta}]$$

$$= M + (\theta-M)\Phi\left(\frac{M-\theta}{r}\right) - r\varphi\left(\frac{M-\theta}{r}\right)$$

$$\mu_o(\theta) = \theta$$

Applying the formulae derived in i) we get by straightforward calculations

$$E[\mu_o^M(\theta)] = \int_{-\infty}^{\infty}\mu_o^M(\theta)\ \frac{1}{s}\ \varphi\left(\frac{\theta-m_o}{s}\right)d\theta$$

$$= M - (M-m_o)\Phi\left(\frac{M-m_o}{\sigma_o}\right) - \sigma_o\varphi\left(\frac{M-m_o}{\sigma_o}\right)$$

$$E[\theta\cdot\mu_o^M(\theta)] = Mm_o + (s^2+m_o^2-Mm_o)\Phi\left(\frac{M-m_o}{\sigma_o}\right) - m_o\sigma_o\varphi\left(\frac{M-m_o}{\sigma_o}\right)$$

Hence

$$\mathrm{Cov}[\mu_o^M(\theta),\mu_o(\theta)] = E[\theta\cdot\mu_o^M(\theta)] - m_o\ E[\mu_o^M(\theta)] = s^2\Phi\left(\frac{M-m_o}{\sigma_o}\right)$$

and

$$\boxed{b_1 = (1-\pi)^2\, w\Phi\left(\frac{M-m_o}{\sigma_o}\right)}\ .$$

As $\mathrm{Cov}[X_1 \wedge M,\ X_2 \wedge M / X_1, X_2 \text{ ordinary}]$

$= \mathrm{Cov}[U_1 \wedge M,\ U_2 \wedge M]$

$= E[\mathrm{Cov}[U_1 \wedge M,\ U_2 \wedge M/_\theta]] + \mathrm{Cov}[E[U_1 \wedge M/_\theta],\ E[U_2 \wedge M/_\theta]]$

$= \mathrm{Var}[\mu_o^M(\theta)]$, we conclude from 19)

$$\boxed{b_2 = (1-\pi)^2\, \mathrm{Cov}[U_1 \wedge M,\ U_2 \wedge M]}\ .$$

To obtain a closed formula for b_3, observe

$$\int_{-\infty}^{M} x(x-\mu)\,\frac{1}{\sigma}\,\varphi(x-\mu)\,dx = -x\sigma\varphi\left(\frac{x-\mu}{\sigma}\right)\Big|_{-\infty}^{M} + \sigma\int_{-\infty}^{M}\varphi\left(\frac{x-m}{\sigma}\right)dx$$

$$= -\sigma M\varphi\left(\frac{M-\mu}{\sigma}\right) + \sigma^2\Phi\left(\frac{M-\mu}{\sigma}\right)$$

$$\int_{-\infty}^{M} x^2\,\frac{1}{\sigma}\,\varphi(x-\mu)\,dx = -\sigma(M+\mu)\,\varphi\left(\frac{M-\mu}{\sigma}\right) + (\mu^2+\sigma^2)\,\Phi\left(\frac{M-\mu}{\sigma}\right)$$

According to 1) the density function of X is

$$f(x) = \int (1-\pi)\,p_o(x/_\theta)\,dU(\theta) + \pi p_e(x) = (1-\pi)\,p_o(x) + \pi p_e(x)$$

with (see 6.1 and 6.2)

$$p_o(x) = \frac{1}{\sigma_o}\,\varphi\left(\frac{x-m_o}{\sigma_o}\right)$$

$$p_e(x) = \frac{1}{\sigma_e}\,\varphi\left(\frac{x-\mu_e}{\sigma_e}\right)$$

Hence

$$\boxed{\begin{aligned} E[(X\wedge M)^2] &= (1-\pi)\left\{(m_o^2+\sigma_o^2)\,\Phi\left(\frac{M-m_o}{\sigma_o}\right)-\sigma_o(M+m_o)\,\varphi\left(\frac{M-m_o}{\sigma_o}\right)\right\} \\ &\quad+\pi\left\{(\mu_e^2+\sigma_e^2)\,\Phi\left(\frac{M-\mu_e}{\sigma_e}\right)-\sigma_e(M+\mu_e)\,\varphi\left(\frac{M-\mu_e}{\sigma_e}\right)\right\} \\ &\quad+M^2\left\{1-(1-\pi)\,\Phi\left(\frac{M-m_o}{\sigma_o}\right)-\pi\Phi\left(\frac{M-\mu_e}{\sigma_e}\right)\right\} \\ &= A \end{aligned}}$$

The same calculations as at the beginning of ii) leading to the formula for $\mu_o^M(\theta)$ are repeated to obtain $E[X\wedge M]$, of course with different parameter values. From this calculation we obtain

$$\boxed{\begin{aligned} E[X\wedge M] &= M+(1-\pi)\left\{(m_o-M)\,\Phi\left(\frac{M-m_o}{\sigma_o}\right)-\sigma_o\varphi\left(\frac{M-m_o}{\sigma_o}\right)\right\} \\ &\quad+\pi\left\{(\mu_e-M)\,\Phi\left(\frac{M-\mu_e}{\sigma_e}\right)-\sigma_e\varphi\left(\frac{M-\mu_e}{\sigma_e}\right)\right\} \\ &= B \end{aligned}}$$

We can now, finally, write

$$\boxed{b_3 = \mathrm{Var}[X\wedge M] = A - B^2}$$

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